

Multisymplectic 3-forms on 7-dimensional manifolds.

Tomáš Salač¹

Abstract: The goal of the paper is to give characterization of closed connected manifolds which admit a global multisymplectic 3-form of some algebraic type. A generic type of such 3-form is equivalent to a G_2 -structure. This is the most interesting case and was solved in [Gr]. Some other algebraic types were solved quite recently. In this paper we give characterization in the remaining cases.

0.1 Introduction

We call a 3-form $\omega \in \Lambda^3((\mathbb{R}^7)^*)$ multisymplectic iff the map

$$\begin{aligned} i_* : \mathbb{R}^7 &\rightarrow \Lambda^2(\mathbb{R}^7)^* \\ v &\mapsto i_v \omega = \omega(v, -, -) \end{aligned} \tag{1}$$

is injective. There are eight orbits of multisymplectic forms under the natural action of $GL(7, \mathbb{R})$ on $\Lambda^3((\mathbb{R}^7)^*)$. We will denote preferred representatives of the orbits by $\omega_i, i = 1, \dots, 8$. Let us denote the stabilizer of ω_i by O_i and a maximal compact subgroup of O_i by K_i . The connected component of identity of K_i is denoted by K_i^0 .

Let M be a 7-dimensional closed connected manifold and let $\rho \in \Omega^3(M)$ be a global 3-form. We say that ρ is of algebraic type $i = 1, \dots, 8$ if $\forall x \in M$ there exists a basis of T_x^*M at the point x such that the 3-form $\rho(x) \in \Lambda^3(T_x^*M)$ belongs to the i -th orbit of the multisymplectic forms. This notion clearly does not depend on a choice of frame. The main goal of this paper is to give topological restrictions on M to admit a global 3-form of a given algebraic type. We will need the following observations.

A global 3-form of the i -th algebraic type on the manifold M is equivalent to a reduction of structure group of the tangent bundle TM of M to K_i . The first goal is to find maximal compact subgroups K_i . The groups O_i were given in [BV]. Without loss of generality we may take $K_i := O_i \cap O(7)$. This is the content of the first part of this paper.

Topological conditions on manifolds which admit a global 3-form of a given algebraic type is given in the second part. Solved cases include the algebraic types 3, 5 and 8. The type 8 is the first solved case and is equivalent to a G_2 -structure. The manifold M admits such structure iff M is orientable and admits a spin structure. This is originally a result of [Gr]. The type 5 was solved in [Le]. It turns out that the case 5 is equivalent to the case 8. We show that the cases 5, 8 are equivalent to the cases 6, 7. We use similar ideas as in the paper [Le]. The type 3 was solved in [D] using techniques introduced in [Th]. We use the same machinery to handle remaining cases with additional assumption on orientability or simple connectedness of manifolds. These are the theorems 9-11. The last section consists of lemma needed in the two first chapters.

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0.1.1 Notation

Notation for the first chapter. We denote the space \mathbb{R}^7 by V and the standard basis by $\{e_1, \dots, e_7\}$. We will denote the dual basis by $\{\alpha_1, \dots, \alpha_7\}$. Let us denote by $\langle v_1, \dots, v_i \rangle$ the linear span of vectors v_1, \dots, v_i . We will denote the stabilizer of the preferred multisymplectic 3-form ω_i by O_i , its maximal compact subgroup by K_i and the connected component of K_i is denoted by K_i^0 .

¹MUÚK, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic, e-mail: salac@karlin.mff.cuni.cz. Research supported by GACR 201/09/H012 and SVV-2011-263317.

Notation for the second chapter. We will denote by ρ^i , resp. η^i , resp. ξ^i a real, resp. a complex, resp. a real trivial bundle of dimension i . We will denote by $w_i(\rho^j)$, resp. $c_i(\eta^j)$, resp. $p_i(\rho^j)$, resp. $e(\rho^j)$ the i -th Stiefel-Whitney, resp. the i -th Chern, resp. the i -th Pontryagin class, resp. the Euler class of the corresponding bundle. We denote null-homotopic maps by $*$.

0.2 Multisymplectic 3-forms.

0.2.1 The 3-form ω_1 .

A representative of the orbit is

$$\omega_1 = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 \wedge \alpha_6. \quad (2)$$

Let us denote by $V_1 = \langle e_7 \rangle$, $V_1^2 = \langle e_3, e_4 \rangle$, $V_2^2 = \langle e_5, e_6 \rangle$, $V_3^2 = \langle e_1, e_2 \rangle$, $W_1^2 = (V_1^2 \oplus V_1)/V_1$, $W_2^2 = (V_2^2 \oplus V_1)/V_1$. Let φ be any element of O_1 . The following statements were proved in [BV].

- φ preserves the subspace V_1 .
- φ induces an automorphism of $W_1^2 \oplus W_2^2$ such that:
 1. $\varphi(W_1^2) = W_1^2, \varphi(W_2^2) = W_2^2$ or
 2. $\varphi(W_1^2) = W_2^2, \varphi(W_2^2) = W_1^2$.

We define a map $sgn : O_1 \rightarrow \mathbb{Z}_2$ by $sgn(\varphi) = 1$ if the first possibility holds and $sgn(\varphi) = -1$ if the latter condition holds. We call $sgn(\varphi)$ the sign of φ . Clearly the map sgn is a group homomorphism.

- The stabilizer O_1 of ω_1 is isomorphic to a semi-direct product

$$(N \times (\mathrm{GL}(W_1^2) \times \mathrm{GL}(W_2^2))) \ltimes \mathbb{Z}_2 \quad (3)$$

such that:

- The first semi-direct product is given by the homomorphism sgn .
- The map $\varphi \in \mathrm{Ker}(sgn) \mapsto \varphi|_{W_1^2 \oplus W_2^2} \in \mathrm{GL}(W_1^2) \times \mathrm{GL}(W_2^2)$ is surjective.
- The group N consists of transformations of the form $Id_V + \varphi_1 + \varphi_2$ where

$$\varphi_1 : V_2^3 \rightarrow V_2^1 \oplus V_2^2 \oplus V_1, \varphi_2 : V_2^1 \oplus V_2^2 \rightarrow V_1.$$

In particular, with respect to usual convention, any element of N is an upper triangular matrix.

Let us define an embedding

$$\begin{aligned} \rho : \mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R}) &\rightarrow \mathrm{GL}(\langle e_1 \rangle) \times \mathrm{GL}(\langle e_2 \rangle) \times \mathrm{GL}(V_1^2) \times \mathrm{GL}(V_2^2) \times \mathrm{GL}(V_1) \\ (\rho(a, b))(v_1, v_2, v_3, v_4, v_5) &= (\det(a^{-1})v_1, \det(b^{-1})v_2, av_3, bv_4, \det(ab)v_5), \end{aligned} \quad (4)$$

where $\mathrm{GL}(2, \mathbb{R})$ acts naturally on $V_1^2 \cong V_2^2 \cong \mathbb{R}^2$. Let us denote the image of $\mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R})$ under ρ by $\mathrm{GL}(2, \mathbb{R})_{2,2}^2$.

Lemma 1. *The group $\mathrm{GL}(2, \mathbb{R})_{2,2}^2$ gives a splitting of $\mathrm{GL}(W_1^2) \times \mathrm{GL}(W_2^2)$ in the formula (3).*

Proof: Clearly $\mathrm{GL}(2, \mathbb{R})_{2,2}^2 \subset O_1$ and $\mathrm{GL}(2, \mathbb{R})_{2,2}^2$ satisfies all conditions to be a splitting of $\mathrm{GL}(W_1^2) \times \mathrm{GL}(W_2^2)$. \square

Now it is easy to find a maximal compact subgroup K_1 . From the lemmas 9., 8. and the description of N given above follows that K_1 is a semi-direct product of a maximal compact subgroup of $\mathrm{GL}(2, \mathbb{R})_{2,2}^2$ and \mathbb{Z}_2 . A maximal compact subgroup of $\mathrm{GL}(2, \mathbb{R})_{2,2}^2$ is the group $O(2) \times O(2)$. A splitting of the group \mathbb{Z}_2 of the homomorphism sgn is given for example by the transformation

$$e_1 \mapsto e_2, e_2 \mapsto e_1, e_3 \mapsto e_5, e_4 \mapsto e_6, e_5 \mapsto e_3, e_6 \mapsto e_4, e_7 \mapsto -e_7 \quad (5)$$

Theorem 1. *A maximal compact subgroup K_1 of O_1 is generated by the subgroup $O(2) \times O(2)$ of the group defined (4) and the transformation given in (5). In particular K_1 is a subgroup of $SO(7)$.*

0.2.2 The 3-form ω_2 .

A representative chosen with respect to a basis $\{e_1, \dots, e_7\}$ in [BV] is

$$\begin{aligned} \omega_2 &= \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 - \alpha_2 \wedge \alpha_3 \wedge \alpha_7 \\ &+ \alpha_3 \wedge \alpha_4 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_6. \end{aligned} \quad (6)$$

Let us work with a basis $\{f_1, \dots, f_7\}$ which is given by

$$\begin{aligned} f_1 &= e_5 + e_6 \\ f_2 &= -e_5 + e_6 \\ f_3 &= -e_5 - e_6 + e_7 \\ \sqrt{2}f_4 &= -e_1 - e_4 \\ \sqrt{2}f_5 &= -e_2 + e_3 \\ \sqrt{2}f_6 &= -e_2 - e_3 \\ \sqrt{2}f_7 &= -e_1 + e_4. \end{aligned}$$

Let us denote by $\{\beta_1, \dots, \beta_7\}$ the dual basis to $\{f_1, \dots, f_7\}$. A new 3-form ω'_2 in the orbit of ω_2 is

$$\begin{aligned} \omega'_2 &= \beta_1 \wedge \beta_4 \wedge \beta_5 - \beta_1 \wedge \beta_6 \wedge \beta_7 + \beta_2 \wedge \beta_5 \wedge \beta_7 \\ &- \beta_2 \wedge \beta_4 \wedge \beta_6 - \beta_3 \wedge \beta_4 \wedge \beta_7 - \beta_3 \wedge \beta_5 \wedge \beta_6. \end{aligned} \quad (7)$$

We will give a description of a maximal compact subgroup of the stabilizer $O_2 := \text{Stab}(\omega'_2)$. Let us denote by $V_3 = \langle f_1, f_2, f_3 \rangle, V_4 = \langle f_4, \dots, f_7 \rangle, W_4 = V/V_3$. We notice that

$$\omega'_2 + \beta_1 \wedge \beta_2 \wedge \beta_3 = \omega'_5. \quad (8)$$

3-form ω'_5 is the 3-form in the orbit of ω_5 given by the formula (41) with respect to the basis $\{i, j, k, e_4, e_4i, e_4j, e_4k | i, j, k \in \tilde{\mathbb{H}}\}$ of $\tilde{\mathbb{O}}$. This leads to the following observation.

Lemma 2. *The group $\tilde{\text{SL}}(2, \mathbb{R})_{3,4}^2$ given in the formula (44) is a subgroup of K_2 .*

Proof: Let $g \in \tilde{\text{SL}}(2, \mathbb{R})_{3,4}^2$. From the formula (44) follows that $g^*(\beta_1 \wedge \beta_2 \wedge \beta_3) = \beta_1 \wedge \beta_2 \wedge \beta_3$. Since $\tilde{\text{SL}}(2, \mathbb{R})_{3,4}^2 \subset \tilde{G}_2$, then $g^*\omega'_5 = \omega'_5$. Thus $g^*\omega'_2 = \omega'_2$. \square

Lemma 3. *There is an isomorphism*

$$O_2 \cong (L \rtimes \tilde{\text{SL}}(2, \mathbb{R})_{3,4}^2) \rtimes \mathbb{R}^* \quad (9)$$

where:

- The group \mathbb{R}^* is the cokernel of O_2 under the homomorphism $g \in O_2 \mapsto \det(g|_{V_3})$.
- The group L consists of endomorphisms of the form $\text{Id}_V + \varphi$ where $\varphi : V_4 \rightarrow V_3$.

Proof: In [BV] is given an isomorphism

$$O_2 \cong ((L \rtimes \text{Spin}(1, 2)) \rtimes \text{SO}(1, 2)) \rtimes \mathbb{R}^*.$$

Thus we have to verify isomorphism between the semi-direct product $\text{Spin}(1, 2) \rtimes \text{SO}(1, 2)$ given in [BV] and the group $\tilde{\text{SL}}(2, \mathbb{R})_{3,4}^2$. We identify $V_4 \cong \tilde{\mathbb{H}}$ by a map

$$f_4 \rightarrow 1_2, f_5 \rightarrow i, f_6 \rightarrow j, f_7 \rightarrow k$$

and $V_3 \cong \text{Im} \tilde{\mathbb{H}}$ by

$$f_1 \rightarrow i, f_2 \rightarrow j, f_3 \rightarrow k,$$

where i, j, k are given in (34) and $1_2 \in \text{GL}(2, \mathbb{R})$ is the identity matrix. This also identifies $V \cong \text{Im} \tilde{\mathbb{O}}$. Let $g \in \tilde{\text{SL}}(2, \mathbb{R})_{3,4}^2$, then:

- A standard quadratic form² of signature $\{+, -, -\}$ on V_3 is invariant under g .
- A standard quadratic form³ of signature $\{+, +, -, -\}$ on V_4 is invariant under g .
- if $g = (1_2, a)$, compare to the formula (44), then g commutes with the image of the map $V_3 \rightarrow \text{End}(W_4)$ induced by ω'_2 .

Thus $\tilde{\text{SL}}(2, \mathbb{R})_{3,4}^2$ satisfies all the conditions which define the semi-direct product $\text{Spin}(1, 2) \rtimes \text{SO}(1, 2)$ inside O_2 . \square

From the lemmas 9., 8. and the description of L given above follows that K_1 is a semi-direct product of a maximal compact subgroup of $\tilde{\text{SL}}(2, \mathbb{R})_{3,4}^2$ and a maximal compact subgroup of \mathbb{R}^* . A maximal compact subgroup $T_{2,2,2}^2$ of $\tilde{\text{SL}}(2, \mathbb{R})_{3,4}^2$ is given (45) with explicit matrix realizations of the connected component of the identity given in (46). The other component of $T_{2,2,2}^2$ contains for example a transformation

$$f_1 \mapsto -f_1, f_2 \mapsto f_2, f_3 \mapsto -f_3, f_4 \mapsto -f_5, f_5 \mapsto -f_4, f_6 \mapsto f_7, f_7 \mapsto f_6 \quad (10)$$

One can verify that a transformation

$$f_1 \mapsto f_1, f_2 \mapsto f_2, f_3 \mapsto -f_3, f_4 \mapsto f_7, f_5 \mapsto f_6, f_6 \mapsto f_5, f_7 \mapsto f_4 \quad (11)$$

gives a splitting of a maximal compact subgroup \mathbb{Z}_2 of \mathbb{R}^* of the map $g \in \text{O}_2 \mapsto \det(g|_{V_3})$. Thus we have the following characterization of K_2 .

Theorem 2. *The group K_2 has four components and is generated by the connected component of the identity K_2^0 given in the formula (46) and the transformations given in (10) and (11).*

0.2.3 The 3-form ω_3 .

A representative is

$$\omega_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3 - \alpha_6 \wedge \alpha_7 + \alpha_4 \wedge \alpha_5). \quad (12)$$

Let us denote by $V_1 = \langle e_1 \rangle, V_6 = \langle e_2, \dots, e_7 \rangle$. Then from [BV] we have that:

- Any element $\varphi \in \text{O}_3$ preserves the subspace V_6 .
- The 3-form ω_3 induces a symplectic structure on V_6 .
- The stabilizer O_3 is isomorphic to a semi-direct product

$$\text{O}_3 \cong \text{N} \ltimes \text{CSp}(3, \mathbb{R}), \quad (13)$$

where

- The group $\text{CSp}(3, \mathbb{R})$ is isomorphic to the cokernel of O_4 under the homomorphism $g \in \text{O}_4 \mapsto g|_{V_6} \in \text{End}(V_6)$.
- The group N consists of the endomorphisms of the form $\text{Id}_V + \varphi$ where $\varphi : V_1 \rightarrow V_6$.

The connected component of the identity K_3^0 is isomorphic to a maximal compact subgroup $\text{U}(3)$ of $\text{Sp}(3, \mathbb{R})$. The other component contains for example the transformation

$$e_1 \mapsto -e_1, e_2 \mapsto e_2, e_3 \mapsto -e_3, e_4 \mapsto e_4, e_5 \mapsto -e_5, e_6 \mapsto e_6, e_7 \mapsto -e_7. \quad (14)$$

Theorem 3. *The group K_2 has two components and is generated by the connected component of the identity which is isomorphic to a maximal compact subgroup $\text{U}(3)$ of $\text{Sp}(3, \mathbb{R})$ and the transformation given in (14).*

²With respect to the basis $\{f_1, f_2, f_3\}$.

³With respect to the basis $\{f_4, \dots, f_7\}$.

0.2.4 The 3-form ω_4 .

A representative of the orbit is

$$\omega_4 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_7) + \alpha_2 \wedge \alpha_3 \wedge \alpha_4. \quad (15)$$

Let us denote by $V_1 = \langle e_1 \rangle$, $V_3 = \langle e_5, e_6, e_7 \rangle$, $U_3 = \langle e_2, e_3, e_4 \rangle$, $V_6 = \langle e_2, \dots, e_7 \rangle$, $W_3 = V_6/V_3$, $W_1 = V/V_6$. Then from [BV] we have that:

- Any element $\varphi \in O_3$ preserves the subspaces V_3, V_6 .
- The 3-form ω_3 induces a symplectic structures on V_6 and a volume form on W_3 .
- The stabilizer is isomorphic to a semi-direct product

$$O_3 \cong (N \ltimes \text{SL}(W_3)) \ltimes \mathbb{R}^*, \quad (16)$$

where

- The group \mathbb{R}^* is isomorphic to the cokernel of O_4 under the homomorphism $\varphi \in O_6 \mapsto (\varphi|_{W_1}) \in \text{End}(W_1)$.
- The restriction to W_3 gives epimorphism

$$\{\varphi \in O_4 \mid \varphi|_{V_1} = \text{Id}_{V_1}\} \rightarrow \text{SL}(W_3). \quad (17)$$

- The group N consists of endomorphisms of the form $\text{Id}_V + \varphi_1 + \varphi_2$ where

$$\varphi_1 : V_1 \rightarrow V_6, \varphi_2 : U_3 \rightarrow V_3$$

Let us define

$$\begin{aligned} \phi : \text{SO}(3) &\rightarrow \text{End}(V_1) \oplus \text{End}(U_3) \oplus \text{End}(V_3) \\ (\phi(A))(v_1, v_2, v_3) &= (v_1, Av_2, Av_3), \end{aligned} \quad (18)$$

where $\text{SO}(3)$ acts naturally on $V_3, U_3 \cong \mathbb{R}^3$ with respect to the preferred basis given above. Let us denote the group $\phi(\text{SO}(3))$ by $\text{SO}(3)_{3,3}$.

Lemma 4. *The group $\text{SO}(3)_{3,3}$ given in (18) is a subgroup of O_4 .*

Proof: Let $g \in \text{SO}(3)_{3,3}$. We have to check that:

- $g^*(\alpha_2 \wedge \alpha_3 \wedge \alpha_4) = \alpha_2 \wedge \alpha_3 \wedge \alpha_4$.
- $g^*(\alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_7) = \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_7$.

The first point is straightforward since g restricted to U_3 is a volume preserving transformation. The second point follows from the following computation. Let us denote by $\omega = \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_7$. Then we can write

$$\omega(-, -) = g(J-, -),$$

where g is the standard scalar product on V_6 with respect to the basis $\{e_2, \dots, e_7\}$ and J is the complex structure on V_6 with the block matrix representation

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where all the block matrices are of rank 3×3 . The lemma follows from the matrix identity $q^T J^T q = J^T$ for any $q \in \text{SO}(3)_{3,3}$. \square

The group $\text{SO}(3)_{3,3}$ gives a splitting of a maximal compact subgroup of (17). From the lemmas 9. 8. and the description of N follows that a maximal compact subgroup K_4 is isomorphic to a semi-direct product $\text{SO}(3)_{3,3} \ltimes \mathbb{Z}_2$. The connected component of the identity is isomorphic to $\text{SO}(3)_{3,3}$. The other component of K_4 contains for example the transformation

$$e_1 \mapsto -e_1, e_2 \mapsto e_2, e_3 \mapsto e_3, e_4 \mapsto e_4, e_5 \mapsto -e_5, e_6 \mapsto -e_6, e_7 \mapsto -e_7. \quad (19)$$

Theorem 4. *The group K_4 has two components and is generated by its connected component of the identity $\text{SO}(3)_{3,3}$ and the transformations given in (19).*

0.2.5 The 3-form ω_5 .

A preferred representative is

$$\begin{aligned}\omega_5 &= \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7 \\ &+ \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6.\end{aligned}\tag{20}$$

Maximal compact is $\mathrm{SO}(4)_{3,4}$ in $\tilde{\mathrm{G}}_2$, see [Le]. The 3-form ω_5 is given by the multiplication of $\tilde{\mathcal{O}} \cong \tilde{\mathbb{H}} \oplus \tilde{\mathbb{H}}$ with respect to the basis

$$e_1 \sim (i, 0), e_2 \sim (0, 1), e_3 \sim (0, i), e_4 \sim (j, 0), e_5 \sim (k, 0), e_6 \sim (0, j), e_7 \sim (0, k).\tag{21}$$

0.2.6 The 3-form ω_6 .

Let us choose a representative

$$\omega_6 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7 - \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 \wedge \alpha_7.\tag{22}$$

A 3-form ω'_6 used in [BV] is related to ω_6 by the formula $\omega'_6 = (g^{-1})^* \omega_6$ where $g : V \rightarrow V$ is given by

$$g(e_3) = -e_7, g(e_7) = e_3, g(e_5) = -e_5, g(e_6) = -e_6, g(e_i) = e_i, i = 1, 2, 4.$$

Let us denote by $V_1 = \langle e_3 \rangle, V_2 = \langle e_1, e_2 \rangle, V_3 = V_1 \oplus V_2, V_4 = \langle e_4, \dots, e_7 \rangle, W_4 = V/V_3$. Let $\mathrm{U}(1)_3$ be the subgroup of diagonal matrices of the unitary group $\mathrm{SU}(2)$ and let $\mathrm{U}(1)_3 \times_{\mathbb{Z}_2} \mathrm{SU}(2)$ be the natural subgroup of $\mathrm{SO}(4)_{3,4} \subset \tilde{\mathrm{G}}_2$ below the formula (42).

Lemma 5. *The group $\mathrm{U}(1)_3 \times_{\mathbb{Z}_2} \mathrm{SU}(2) = \{g \in \mathrm{SO}(4)_{3,4} | g^*(\alpha_3) = \alpha_3\}$ is a subgroup of K_6 .*

Proof: We have that $\omega_6 - \alpha_3 \wedge (\alpha_4 \wedge \alpha_7 - \alpha_5 \wedge \alpha_6) = \tilde{\omega}_5$, where $\tilde{\omega}_5$ is the 3-form in the orbit of ω_5 given by the formula (41) with respect to the basis $\{i, j, k, e_4, ie_4, je_4, ke_4 | i, j, k \in \mathbb{H}\}$ of $\tilde{\mathcal{O}}$ as in the formula (37). Let $g \in \mathrm{SO}(4)_{3,4}$, then $g^*(\tilde{\omega}_6) = \tilde{\omega}_6$ and $g^*(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) = \alpha_1 \wedge \alpha_2 \wedge \alpha_3$. Since $g^*(\alpha_3) = \alpha_3$, we have that $g(V_2) = V_2$ and that

$$\begin{aligned}g^*(\tilde{\omega}_6) &= g^*(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7 \\ &- \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 \wedge \alpha_7 - \alpha_3 \wedge \alpha_4 \wedge \alpha_7 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6) \\ &= \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + g^*(-\alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7 \\ &- \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_2 \wedge \alpha_5 \wedge \alpha_7) - \alpha_3 \wedge g^*(\alpha_4 \wedge \alpha_7 - \alpha_5 \wedge \alpha_6) \\ &= \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 - \alpha_3 \wedge g^*(\alpha_4 \wedge \alpha_7 - \alpha_5 \wedge \alpha_6)\end{aligned}$$

for some two forms $\beta_1, \beta_2 \in \Lambda^2 V_4$. Comparing this to $g^*(\tilde{\omega}_6) = \tilde{\omega}_6$ we obtain that

$$\begin{aligned}\beta_1 &= -\alpha_4 \wedge \alpha_5 + \alpha_6 \wedge \alpha_7 \\ \beta_2 &= -\alpha_4 \wedge \alpha_6 - \alpha_5 \wedge \alpha_7 \\ g^*(\alpha_4 \wedge \alpha_7 - \alpha_5 \wedge \alpha_6) &= \alpha_4 \wedge \alpha_7 - \alpha_5 \wedge \alpha_6.\end{aligned}$$

This proves that $g^*(\omega_6) = \omega_6$. \square

Lemma 6. *The connected component of K_6 is isomorphic to $\mathrm{U}(1)_3 \times_{\mathbb{Z}_2} \mathrm{SU}(2)$.*

Proof: From [BV] we have an isomorphism

$$\mathrm{O}_6 \cong ((\mathrm{L} \rtimes \mathrm{SL}(2, \mathbb{C})) \rtimes \mathrm{SO}(2)) \rtimes \mathbb{R}^*\tag{23}$$

where:

- the group \mathbb{R}^* is the cokernel of O_6 under the homomorphism $g \in \mathrm{O}_6 \mapsto \det(g|_{V_2})$.

- the group L consists of transformations of the form $Id + \varphi_1 + \varphi_2$ where

$$\varphi_1 : V_2 \rightarrow (V_4 \oplus V_1), \varphi_2 : V_4 \rightarrow V_1. \quad (24)$$

We have that the group $U(1)_3 \times_{\mathbb{Z}_2} SU(2)$ satisfy the following conditions:

- The subgroup $U(1)_3$ operates with respect to the standard scalar product on V_2 as orthogonal transformations.
- The subgroup $SU(2)$ commutes with the image of the map $\lambda : V_3 \rightarrow End(W_4)$ given in [BV].

Comparing this to [BV], we deduce that $U(1)_3 \times_{\mathbb{Z}_2} SU(2)$ is a splitting of a maximal compact subgroup of $SL(2, \mathbb{C}) \rtimes SO(2)$. The rest is a consequence of the lemma 9, description of L and the formula (23). \square

From the lemma 9. follows that $K_6 \cong (U(1)_3 \times_{\mathbb{Z}_2} SU(2)) \rtimes \mathbb{Z}_2$. Thus to complete the description of K_6 , we have to find an element $g \in K_6$ which does not belong to the connected component of the identity. Such element is for example the transformation

$$e_1 \mapsto e_2, e_2 \mapsto e_1, e_3 \mapsto -e_3, e_4 \mapsto e_4, e_5 \mapsto e_6, e_6 \mapsto e_5, e_7 \mapsto -e_7. \quad (25)$$

Thus we can formulate the following theorem.

Theorem 5. *The group K_6 is generated by the connected component of the identity which is equal to $U(1)_3 \times_{\mathbb{Z}_2} SU(2)$ defined in the lemma 5. and the transformation (25). In particular $K_6 \subset SO(4)_{3,4}$.*

0.2.7 The 3-form ω_7 .

Let us choose a 3-form

$$\begin{aligned} \omega_7 = & \alpha_1 \wedge \alpha_4 \wedge \alpha_5 - \alpha_1 \wedge \alpha_6 \wedge \alpha_7 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 \\ & + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6. \end{aligned} \quad (26)$$

The 3-form ω'_7 used in [BV] is related to ω_7 by the formula $\omega'_7 = (g^{-1})^* \omega_7$ where $g : V \rightarrow V$ is given by

$$g(e_1) = -e_4, g(e_2) = -e_7, g(e_3) = e_5, g(e_4) = -e_6, g(e_5) = e_3, g(e_6) = -e_1, g(e_7) = e_2.$$

Let us denote by V_3 the span $\langle e_1, e_2, e_3 \rangle$ and by V_4 the span $\langle e_4, \dots, e_7 \rangle$. Let us denote by $W_4 = V/V_3$.

Theorem 6. $SO(4)_{3,4} = \{g \in G_2 | g^*(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) = \alpha_1 \wedge \alpha_2 \wedge \alpha_3\}$ is a maximal compact subgroup of K_7 .

Proof: We notice that

$$\omega_8 = \omega_7 + \alpha_1 \wedge \alpha_2 \wedge \alpha_3, \quad (27)$$

where ω_8 is the 3-form (29). This implies that $SO(4)_{3,4} \subset K_7$. We now show also the opposite inclusion. The equation (27) naturally leads to an isomorphism $V \cong Im \mathbb{H} \oplus \mathbb{H} \cong I, \mathbb{O}$ given by

$$e_1 \mapsto (i, 0), e_2 \mapsto (0, 1), e_3 \mapsto (0, i), e_4 \mapsto (j, 0), e_5 \mapsto (k, 0), e_6 \mapsto (0, j), e_7 \mapsto (0, k). \quad (28)$$

This implies that

- The action of $SO(4)_{3,4}$ restricted to V_3 is orthogonal with respect to the standard scalar product.
- The action of the subgroup $1 \times SU(2) \subset SO(4)_{3,4}$ on V_4 commutes with the image of $\lambda : V_3 \rightarrow End(W_4)$ introduced in [BV].

This observation together with the lemma (9) and [BV] gives that $SO(4)_{3,4}$ is a splitting of a maximal compact subgroup of O_7 . \square

0.2.8 The 3-form ω_8 .

This is a well known case. A representative is

$$\begin{aligned}\omega_8 = & \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 - \alpha_1 \wedge \alpha_6 \wedge \alpha_7 \\ & + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6.\end{aligned}\tag{29}$$

The stabilizer is the exceptional compact 14 dimensional Lie group $O_8 \cong G_2$. The 3-form ω_8 is given by the standard multiplication table given by the Cayley-Dickinson construction.

0.3 Manifolds admitting a global multisymplectic 3-form of one type.

We will first consider the global 3-forms of type 5, 6, 7, 8. The known implications in the following theorem are $1 \leftrightarrow 2$, which is result of [Gr], and $2 \leftrightarrow 3$ which was proved in [Le]. We use similar argument as in [Le] to prove also the remaining implications.

The case 3 was solved in [D]. The method used in [D] is based on computing Postnikov invariants. The method is explained in [Th]. We use the same approach to handle the remaining cases.

0.3.1 Global 3-forms of type $\omega_5, \omega_6, \omega_7, \omega_8$.

Theorem 7. *Let M be a closed connected 7-dimensional manifold. Then the following are equivalent:*

1. M is orientable and spinnable.
2. M admits a global 3-form of algebraic type 8.
3. M admits a global 3-form of algebraic type 5.
4. M admits a global 3-form of algebraic type 6.
5. M admits a global 3-form of algebraic type 7.

Proof: $(1) \leftrightarrow (2)$ is given in [Gr]. $(2) \leftrightarrow (3)$ is given in [Le].

$(2) \leftrightarrow (4) \leftrightarrow (5)$. From the theorem 12. and the remark 1. given below the theorem follows that a G_2 -structure implies $SU(2)_{0,4}$ -structure where $SU(2)_{0,4}$ is the subgroup $SU(2) \cong 1 \times SU(2) \subset SO(4)_{3,4}$ where $SO(4)_{3,4}$ is given by the formula (42). Thus G_2 -structure implies also reduction to all subgroups which contain $SU(2)_{0,4}$. We have proved that $SU(2)_{0,4} \leq K_7, K_6 \leq G_2$. The rest follows from the theorem 13. \square

0.3.2 A global 3-form of type ω_4 .

We will consider only the orientable case. We first prove the following observation.

Lemma 7. *Let M be an orientable 7-dimensional closed connected manifold such that M admits a global 3-form of type 4. Then M admits a reduction to $U(1)$.*

Proof: By assumption M admits a reduction to $K_4 \cap SO(7) = SO(3)_{3,3}$. The formula (18) implies that the tangent bundle TM decomposes as $TM \cong \xi^1 \oplus \rho_1^3 \oplus \rho_2^3$, where ρ_1^3 and ρ_2^3 are isomorphic real orientable 3-dimensional vector bundles. Moreover we can choose a complex structure J on $\rho_1^3 \oplus \rho_2^3$ such that $J(\rho_1^3) = \rho_2^3$ and $J(\rho_2^3) = \rho_1^3$. This implies that $w_2(M) = w_2(\rho_1^3) + w_2(\rho_2^3)$ is zero and thus the manifold M admits a $SU(2)$ -structure, see the theorem 7. and the lemma 12. Since the Chern classes of a complex vector bundle do not depend on a choice of

a complex structure, see [MS], the first Chern class of the complex bundle $(\rho_1^3 \oplus \rho_2^3)^4$ is trivial and thus the bundle admits a $SU(3)$ -structure. By a result of Thomas, see [T], M has two everywhere linearly independent sections and thus the bundle $(\rho_1^3 \oplus \rho_2^3)$ has a nowhere zero section ζ . Let us denote by $\xi^2 = \langle \zeta, J\zeta \rangle$. Then $\rho_1^3 \oplus \rho_2^3 \cong \xi^2 \oplus \eta^2$ where η^2 is a complex 2-dimensional bundle with $SU(2)$ -structure and the with complex structure $J|_{\eta^2}$. Let us denote $\eta_i^1 := \eta^2 \cap \rho_i^3$ for $i = 1, 2$. Then $\eta_1^1 \oplus \eta_2^1 = \eta^2$ is a splitting of η^2 into two complex line bundles. Then $c_1(\eta_1^1) = -c_1(\eta_2^1)$ and in particular J is a complex anti-linear isomorphism of these two line bundles. \square

Let us denote the subgroup in the lemma 7. isomorphic to $U(1)$ by L_4 . An inclusion $L_4 \hookrightarrow SO(7)$ factors through $L_4 \hookrightarrow SU(2) \hookrightarrow SO(7)$. The long exact sequence for the homotopy groups of the principal fibration $L_4 \rightarrow SO(7) \rightarrow Q_4$ yields that the homotopy groups of $Q_4 = SO(7)/L_4$ are $\pi_1(Q_4) \cong \mathbb{Z}_2, \pi_2(Q_4) \cong \mathbb{Z} \cong \pi_3(Q_4)$ and are trivial for $7 > i > 3$ and $i = 0$. Thus from the lemma 11. follows that it is sufficient to find a lift of $f : M \rightarrow BSO(7)$ to BL_4 over a 4-skeleton M^4 of M where f is the tautological map.

Let us factor the fibration $p : BL_4 \rightarrow BSO(7)$ into fibrations $BL_4 \rightarrow BSU(2) \rightarrow BG_2 \rightarrow BSO(7)$. Recall that there exist a lift of f to $BSU(2)$ iff $w_2(M) = 0$. Let us build a Postnikov tower for a fibration $q : BL_4 \rightarrow BSU(2)$. We have that $SU(2)/L_4 \cong S^2$ and thus we have

$$\begin{array}{ccccc}
 S^2 & \longrightarrow & K(\mathbb{Z}, 2) & & \\
 \downarrow & & \downarrow & & \\
 BL_4 & \xrightarrow{q \times \alpha} & BSU(2) \times K(\mathbb{Z}, 2) & \xrightarrow{c_2 \otimes 1 - 1 \otimes f^2} & K(\mathbb{Z}, 4) \\
 & \searrow q & \downarrow 1 \times * & & \\
 & & BSU(2) & \xrightarrow{*} & K(\mathbb{Z}, 3)
 \end{array} \quad . \tag{30}$$

Here the map h has to be a homotopy equivalence $BL_4 \cong K(\mathbb{Z}, 2)$ such that $h^*(\alpha) = c_1$ where $\alpha \in H^2(K(\mathbb{Z}, 2), \mathbb{Z})$ and $c_1 \in H^2(L_4, \mathbb{Z})$ are generators. We may choose c_1 such that $q^*(c_2) = -c_1^2$ where $c_2 \in H^4(BSU(2), \mathbb{Z})$ is the second Chern class of the tautological bundle. Thus the second Postnikov invariant is $c_2 \otimes 1 - 1 \otimes c_1^2$.

Theorem 8. *Let M be a closed orientable manifold. Then M admits a global 3-form of algebraic type 4 iff $w_2(M) = 0$ and there exist a class $e \in H^2(M, \mathbb{Z})$ such that $e^2 = \frac{1}{2}p_1(M)$.*

Proof: Let us assume that $w_2(M) = 0$ and that M is orientable. Then the tangent bundle TM of M is isomorphic to $\eta^4 \oplus \xi^3$ where ξ^3 is trivial 3-dimensional bundle and η^4 has a $SU(2)$ -structure. Then $-2c_2(\eta^4) = p_1(M)$. As we have argued above, the bundle η^4 decomposes as a sum of two complex line bundles iff $\exists e \in H^2(M, \mathbb{Z})$ such that $-e^2 = c_2(\eta^4)$. \square

0.3.3 A global 3-form of type ω_3 .

This was solved in [D].

Theorem 9. *Let M be an orientable closed connected 7-dimensional manifold. Then M admits a global 3-form of type 3 iff $\beta(w_2(M)) = 0$.*

0.3.4 A global 3-form of type ω_2 .

We will consider reduction to the connected component of the identity of K_2 . Explicit realization of K_2^0 is given in the formula (46). In particular we see that BK_2^0 is an Eilenberg-MacLane space $K(\mathbb{Z} \times \mathbb{Z}, 2)$. In particular $H^2(BK_2^0, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$ and is generated by the Euler classes of two 2-dimensional bundles of the tautological bundle corresponding to the subgroups $(\theta)_2, (\rho)_2$ in the formula (46). Let us denote these generators by α, β .

⁴Recall that $SO(3)_{3,3}$ sits in $U(3)$.

The homotopy groups $\pi_i(\mathrm{SO}(7)/\mathrm{K}_2^0)$ are zero for $4 < i < 7$. From the lemma 11., as in the case of ω_4 , follows that it suffices to consider Postnikov invariants up to dimension four. The group K_2^0 is a subgroup of G_2 and thus the necessary condition is $w_2(M) = 0$. The group K_2^0 is also a subgroup of $\mathrm{SU}(3)$. Let us denote by Q_2 the homogeneous space $Q_2 := \mathrm{SU}(3)/\mathrm{K}_2^0$. Then $\pi_1(Q_2) \cong \pi_0(Q_2) = 0, \pi_2(Q_2) = \mathbb{Z} \times \mathbb{Z}, \pi_3(Q_2) \cong \mathbb{Z}$. We consider a Postnikov tower

$$\begin{array}{ccccc}
 Q_2 & \longrightarrow & K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) & & (31) \\
 \downarrow & & \downarrow & & \\
 \mathrm{BK}_2^0 & \xrightarrow{q \times \alpha \times \beta} & \mathrm{BSU}(3) \times K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) & \xrightarrow{k_2} & K(\mathbb{Z}, 4) \\
 & \searrow q & \downarrow 1 \times * \times * & & \\
 & & \mathrm{BSU}(3) & \xrightarrow{*} & K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 3)
 \end{array}$$

for the fibration $Q_2 \rightarrow \mathrm{BK}_2^0 \rightarrow \mathrm{BSU}(3)$.

Theorem 10. *Let M be a 7-dimensional closed simply connected and connected manifold. Then M admits a global 3-form of type ω_2 iff $w_2(M) = 0$ and there exist classes $e, f \in H^2(M, \mathbb{Z})$ such that $e^2 + f^2 + ef = \frac{1}{2}p_1(M)$.*

Proof: First we easily find that $q^* : H^4(\mathrm{BSU}(3), \mathbb{Z}) \rightarrow H^4(\mathrm{BK}_2^0, \mathbb{Z})$ is given by $q^*(c_2) = -(\alpha^2 + \beta^2 + \alpha\beta)$. This implies that Postnikov invariant k_2 is equal to $c_2 + \alpha^2 + \beta^2 + \alpha\beta$. The map $H^4(\mathrm{BSO}(7), \mathbb{Z}) \rightarrow H^4(\mathrm{BSU}(3), \mathbb{Z})$ is given $p_1 \mapsto -2c_2$. This gives the theorem. \square

0.3.5 A global 3-form of type ω_1 .

We will consider reduction to the connected component of the identity of K_1 . The connected component of the identity K_1^0 is isomorphic to the two dimensional torus T^2 . In particular BK_1^0 is an Eilenberg-MacLane space $K(\mathbb{Z} \times \mathbb{Z}, 2)$. Let us denote by α, β the Euler classes of the two 2-dimensional vector bundles of the tautological bundle. The non-trivial i -th homotopy groups for $i < 7$ of the quotient $Q_1 := \mathrm{SO}(7)/\mathrm{K}_1^0$ are $i = 3, 4$, i.e. $\pi_3(Q_1) \cong \mathbb{Z} \times \mathbb{Z}, \pi_4(Q_1) \cong \mathbb{Z}$. The lemma 11. implies that we may consider Postnikov invariants of dimension equal or smaller to four. We consider a Postnikov tower

$$\begin{array}{ccccc}
 Q_1^1 & & & & (32) \\
 \downarrow & & & & \\
 Q_1 & \longrightarrow & K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) & & \\
 \downarrow & & \downarrow i & & \\
 \mathrm{BK}_1^0 & \xrightarrow{q_1} & E_0 & \xrightarrow{k_2} & K(\mathbb{Z}, 4) \\
 & \searrow q & \downarrow p & & \\
 & & \mathrm{BSO}(7) & \xrightarrow{1 \times W_3 + W_3 \times 1} & K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 3)
 \end{array}$$

for the fibration $Q_1 \rightarrow \mathrm{BK}_1^0 \rightarrow \mathrm{BSO}(7)$, where W_3 is the generator of $H^3(\mathrm{BSO}(7), \mathbb{Z}) \cong \mathbb{Z}_2$.

Theorem 11. *Let M be a 7-dimensional closed simply connected and connected manifold. Then M admits a global 3-form of type 1 iff there exist classes $e, f \in H^2(M, \mathbb{Z})$ such that $\rho_2(e + f) = w_2(M)$ and $e^2 + f^2 = p_1(M)$.*

Proof: The Serre sequence gives exact sequences

$$\begin{aligned}
 0 &\rightarrow H^2(E_0, \mathbb{Z}) \rightarrow H^2(\mathrm{BK}_1^0, \mathbb{Z}) \rightarrow 0 \\
 0 &\rightarrow H^2(E_0, \mathbb{Z}) \rightarrow H^2(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2), \mathbb{Z}) \rightarrow H^3(\mathrm{BSO}(7)) \rightarrow 0
 \end{aligned}$$

In particular we may view α, β as generators of $H^2(E_0, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$ and up to a change of basis $\{e_1, e_2\}$ of $H^2(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2), \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$ we may also assume that $i^*\alpha = 2e_1, i^*\beta = e_2$. We also get a diagram

$$\begin{array}{ccccccc} H^4(\text{BSO}(7), \mathbb{Z}) & \xrightarrow{p^*} & H^4(E_0, \mathbb{Z}) & \xrightarrow{i^*} & H^4(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2), \mathbb{Z}) \\ & & \downarrow \cong & & \\ 0 \rightarrow H^3(Q_1^1, \mathbb{Z}) & \xrightarrow{\tau} & H^4(E_0, \mathbb{Z}) & \xrightarrow{q_1^*} & H^4(\text{BK}_1^0, \mathbb{Z}) \rightarrow 0, \end{array}$$

where the bottom row is exact and the upper row is a complex. In particular we obtain that $H^4(E_0, \mathbb{Z})$ is a free abelian group isomorphic to \mathbb{Z}^4 which is generated by $k_2, \alpha^2, \alpha\beta, \beta^2$. Now $q^*(p_1) = \alpha^2 + \beta^2 \in H^4(\text{BK}_1^0, \mathbb{Z})$ where $p_1 \in H^4(\text{BSO}(7), \mathbb{Z})$ is the first Pontryagin class which is a generator of $H^4(\text{BSO}(7), \mathbb{Z})$. Thus there exists an integer $a \in \mathbb{Z}$ such that $p_1 = ak_2 - \alpha^2 - \beta^2$. This implies that $q_1^*(ak_2) = 4e_1^2 + e_2^2$. But this implies that $a = \pm 1$. \square

0.4 Appendix.

0.4.1 Algebras.

Let us recall the Cayley-Dickinson construction of $*$ -algebras. Let $(A, *)$ be a $*$ -algebra. We write the conjugation as $a \mapsto *(a) = \bar{a}$ for $a \in A$. If A is an $*$ -algebra, we define a new $*$ -algebra $CD(A)$ such that $CD(A) \cong A \oplus A$ as a vector space with the multiplication and conjugation given by

$$\begin{aligned} (a, b) \cdot (c, d) &= (ac - \bar{d}b, b\bar{c} + ad) \\ \overline{(a, b)} &= (\bar{a}, -b). \end{aligned} \tag{33}$$

For example $CD(\mathbb{R}) = \mathbb{C}$, $CD(\mathbb{C}) = \mathbb{H}$, $CD(\mathbb{H}) = \mathbb{O}$, $CD(\tilde{\mathbb{H}}) = \tilde{\mathbb{O}}$ where \mathbb{R} , resp. \mathbb{C} , resp. \mathbb{H} , resp. $\tilde{\mathbb{H}}$, resp. \mathbb{O} , resp. $\tilde{\mathbb{O}}$ denote the real numbers, resp. the complex numbers, resp. quaternions, resp. pseudo-quaternions, resp. octonions, resp. pseudo-octonions.

Pseudo-quaternions. The algebra $\tilde{\mathbb{H}}$ of pseudo-quaternions is isomorphic to the algebra $M(2, \mathbb{R})$ of real 2×2 matrices. Let us use the following notation

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, ij = k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{34}$$

Let us denote by 1_2 the identity 2×2 -matrix. Let us denote by $\tilde{SL}(2, \mathbb{R}) = \{g \in GL(2, \mathbb{R}) | \det(g) = \pm 1\}$. If $g \in \tilde{SL}(2, \mathbb{R})$, then the conjugation by g is an automorphism of the algebra $\tilde{\mathbb{H}}$.

Octonions. A choice of a subalgebra A of \mathbb{O} isomorphic to \mathbb{H} and a vector e in the orthogonal complement A^\perp with respect to the standard scalar product on \mathbb{O} gives isomorphism of \mathbb{O} with the algebra $\mathbb{H} \oplus \mathbb{H}$ via

$$(a + be) \in \mathbb{O} \leftrightarrow (a, b) \in \mathbb{H} \oplus \mathbb{H}, \tag{35}$$

with the multiplication and the conjugation given by the same formulas as in (33). Alternatively we may write octonions as pairs of quaternions $(p, q) \leftrightarrow p + eq$. In this case the multiplication is given by

$$(a, b)(c, d) = (ac - \bar{d}b, cb + \bar{a}d). \tag{36}$$

Pseudo-octonions. A choice of a subalgebra A of $\tilde{\mathbb{O}}$ isomorphic to \mathbb{H} and a vector e in the orthogonal complement A^\perp with respect to the standard scalar product on $\tilde{\mathbb{O}}$ gives isomorphism of \mathbb{O} with the algebra $\mathbb{H} \oplus \mathbb{H}$ via

$$(a + be) \in \tilde{\mathbb{O}} \leftrightarrow (a, b) \in \mathbb{H} \oplus \mathbb{H}, \quad (37)$$

with the multiplication

$$(a, b)(c, d) = (ac + d\bar{b}, cb + \bar{a}d). \quad (38)$$

If we choose a subalgebra A of $\tilde{\mathbb{O}}$ isomorphic to $\tilde{\mathbb{H}}$ and e as above, then we can identify

$$\begin{aligned} \tilde{\mathbb{H}} \oplus \tilde{\mathbb{H}} &\cong \tilde{\mathbb{O}} \\ (p, q) &\mapsto p + eq. \end{aligned} \quad (39)$$

0.4.2 3-forms from algebras.

Let g , resp \tilde{g} be the standard scalar product on \mathbb{O} , resp. $\tilde{\mathbb{O}}$. Recall that the quadratic form associated to the standard scalar product is positive definite in the case of octonions and of signature $(3, 4)$ in the case of pseudo-octonions. Then the formula

$$\omega(a, b, c) = g(ab, c) \quad (40)$$

$$\tilde{\omega}(e, f, g) = \tilde{g}(ef, g) \quad (41)$$

for $a, b, c \in \mathbb{O}$ or $e, f, g \in \tilde{\mathbb{O}}$ defines a 3-form. In particular ω belongs to the orbit $[\omega_8]$ while $\tilde{\omega}$ belongs to the orbit $[\omega_5]$.

0.4.3 Subgroups of the exceptional groups G_2 and \tilde{G}_2 .

Let us recall that the exceptional Lie group G_2 is the group of automorphisms of the algebra \mathbb{O} . Let us denote the group of automorphisms of the algebra $\tilde{\mathbb{O}}$ by \tilde{G}_2 .

Subgroup $SO(4)_{3,4}$ in G_2 and in \tilde{G}_2 . Let us define an embedding of $SO(4)$ in $End(\mathbb{H}) \times End(\mathbb{H})$ by the formula

$$\begin{aligned} \phi : SU(2) \times_{\mathbb{Z}_2} SU(2) &\rightarrow End(\mathbb{H}) \times End(\mathbb{H}) \\ (\phi(a, b))(p, q) &= (apa^{-1}, bqa^{-1}), \end{aligned} \quad (42)$$

where $(p, q) \in \mathbb{H} \oplus \mathbb{H}$. Let us denote the image of ϕ by $SO(4)_{3,4}$. It is shown in [Y] that $SO(4)_{3,4} \subset G_2$ where we identify \mathbb{O} with $\mathbb{H} \oplus \mathbb{H}$ as in (35).

If we identify $\tilde{\mathbb{O}}$ with $\mathbb{H} \oplus \mathbb{H}$ as in the formula (37), then the formula (42) gives an embedding of $SO(4)$ in \tilde{G}_2 . We denote the image of this embedding also as $SO(4)_{3,4}$.

Subgroup $\tilde{SL}(2, \mathbb{R})_{3,4}^2$ of \tilde{G}_2 . Let us denote by $\tilde{SL}(2, \mathbb{R}) = \{a \in GL(2, \mathbb{R}) | \det(a) = \pm 1\}$. Let us denote by

$$\tilde{SL}(2, \mathbb{R})_{3,4}^2 := \{(a, b) | a, b \in \tilde{SL}(2, \mathbb{R}), \det(ab) = 1\} / \mathbb{Z}_2, \quad (43)$$

where $\mathbb{Z}_2 = \{\pm(1_2, 1_2)\}$ and 1_2 is the identity 2×2 -matrix. We embed $\tilde{SL}(2, \mathbb{R})_{3,4}^2$ in $End(\tilde{\mathbb{H}}) \times End(\tilde{\mathbb{H}})$ by the formula

$$(a, b).(p, q) := (apa^{-1}, aqb^{-1}), \quad (44)$$

with $a, b \in \tilde{SL}(2, \mathbb{R})$ and $(p, q) \in \tilde{\mathbb{H}} \oplus \tilde{\mathbb{H}}$. As in (42), one can easily verify that $\tilde{SL}(2, \mathbb{R})_{3,4}^2$ is a subgroup of \tilde{G}_2 where we identify $\tilde{\mathbb{O}}$ with $\tilde{\mathbb{H}} \oplus \tilde{\mathbb{H}}$ as in (39).

From the lemma (9) follows that the group

$$S(O(2) \times_{\mathbb{Z}_2} O(2)) := \{(a, b) | a, b \in O(2), \det(ab) = 1\} \subset \tilde{SL}(2, \mathbb{R})_{3,4}^2 \quad (45)$$

is a maximal compact subgroup of $\tilde{\text{SL}}(2, \mathbb{R})_{3,4}^2$. We denote this group by $\text{T}_{2,2,2}^2$. The connected component of the identity of $\text{T}_{2,2,2}^2$ is realized with respect to the standard basis of $\text{Im}(\tilde{\mathbb{O}})$ by matrices

$$\begin{pmatrix} 1_1 & 0 & 0 & 0 \\ 0 & (-2\alpha)_2 & 0 & 0 \\ 0 & 0 & (\alpha + \beta)_2 & 0 \\ 0 & 0 & 0 & (\alpha - \beta)_2 \end{pmatrix} = \begin{pmatrix} 1_1 & 0 & 0 & 0 \\ 0 & (-\theta - \rho)_2 & 0 & 0 \\ 0 & 0 & (\theta)_2 & 0 \\ 0 & 0 & 0 & (\rho)_2 \end{pmatrix}, \quad (46)$$

where $(\gamma)_2 \in \text{SO}(2)$ denotes rotation by the angle $\gamma \in \mathbb{R}$ and $\alpha + \beta = \theta, \alpha - \beta = \rho$. In particular the connected component of $\text{T}_{2,2,2}^2$ is isomorphic to $\text{SO}(2) \times \text{SO}(2)$.

0.4.4 Semi-direct product of Groups.

Lemma 8. *Let $N \subset \text{GL}(n, \mathbb{R})$ be a closed subgroup. Suppose that N consists of matrices of the form $1_n + A$ where 1_n is the identity matrix and A is a strictly upper (lower) triangular matrix. Then the only subgroup N' of N for which there exists an N' -invariant scalar product on \mathbb{R}^n is the trivial subgroup. Thus the maximal compact subgroup of N is the trivial subgroup.*

Proof: Easy exercise. \square .

Lemma 9. *Let $G_0 = G_1 \rtimes G_2$ be a semi-direct product of Lie groups. Let $K_0 \subset G_0$ be a maximal compact subgroup. Then $K_1 \rtimes K_2$ is a maximal compact subgroup of G_0 where $K_1 = K_0 \cap G_1$, resp. $K_2 = K_0 \cap G_2$ is a maximal compact subgroup of G_1 , resp. G_2 .*

Proof: Clearly $K_i \subset G_i, i = 1, 2$ is a maximal compact subgroup and thus also their semi-direct product is compact. Suppose that K_3 is a compact subgroup of G_0 such that $K_1 \rtimes K_2 \subset K_3$. Then $K_3/(K_3 \cap G_1) \subset K_2$ and also $K_3 \cap G_1 \subset K_1$ but this implies that $K_3 \subset K_1 \rtimes K_2$. \square

0.4.5 More on G_2 -structure.

A G_2 -structure on a closed connected 7-dimensional Riemannian manifold M is a reduction of structure group of the tangent bundle TM of M to G_2 , i.e. it is given by a following commutative diagram of principle bundles

$$\begin{array}{ccccc} G_2 & \longrightarrow & \mathcal{Q} & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \text{Id} \\ \text{SO}(7) & \longrightarrow & \mathcal{P} & \longrightarrow & M \end{array} \quad (47)$$

where \mathcal{P} is the principle bundle of orthonormal frames with respect to some Riemannian metric on M . It is a well known fact that existence of a reduction to G_2 is equivalent to vanishing of the second Stiefel-Whitney class w_2 of TM , see [Gr].

Octonionic structure. An octonionic structure on a 7-dimensional Riemannian manifold (M, g) is a smooth bundle map

$$\mu : (TM \oplus R) \otimes (TM \oplus R) \rightarrow (TM \oplus R), \quad (48)$$

where R is a trivial real line bundle over M such that $\forall x \in M$ there exists an algebra isomorphism between (\mathbb{O}, \cdot) and $(\mathbb{R}_x \oplus T_x M, \mu_x)$ compatible with the metric structures.

Lemma 10. *Let M be a 7-dimensional closed connected manifold. Then the following are equivalent:*

1. *there exists a G_2 -structure on M .*
2. *there exists an octonionic structure in the fibers of TM .*

Proof: See [D].

In the paper [FKMS] can be found the following theorem.

Theorem 12. *Let M be a 7-dimensional closed connected manifold. Then following are equivalent:*

1. *there exists a G_2 -structure on M .*
2. *there exists a $SU(2)$ -structure on M , i.e. $TM \cong \xi^3 \oplus \eta^2$ where ξ^3 is a trivial real 3-dimensional bundle and η^2 is a complex 2-dimensional bundle with $SU(2)$ -structure.*

Proof: See [FKMS]. The assumption on compactness of M is not necessary. \square

Remark 1. *A $SU(2)$ -structure coming from a G_2 -structure can be viewed in the following way. By a result from [T], any orientable 7-dimensional manifold admits two everywhere linearly independent vector fields. Let us denote them ζ_1, ζ_2 . We may assume that ζ_1, ζ_2 are orthonormal. Since G_2 -structure is equivalent to an octonionic structure, the vector field $\zeta_1 \cdot \zeta_2 = \zeta_3$ is also of unit length and orthogonal to ζ_1, ζ_2 . The vector fields $\{\zeta_1, \zeta_2, \zeta_3\}$ span a 3-dimensional vector subbundle ξ^3 of the tangent bundle. The orthogonal complement to ξ^3 is a four dimensional real vector bundle. The multiplication by $\zeta_1, \zeta_2, \zeta_3$ gives quaternionic structure on this bundle and thus also reduction to $SU(2)$.*

0.4.6 Extensions.

Theorem 13. *Let G be a topological group and let $H \leq G$ be a closed subgroup. Let \mathcal{P} be a principal H -bundle over a topological space X . Then there exists a principal G -bundle \mathcal{P}' over X such that the diagram*

$$\begin{array}{ccccc} H & \longrightarrow & \mathcal{P} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow Id \\ G & \longrightarrow & \mathcal{P}' & \longrightarrow & X \end{array} \quad (49)$$

commutes.

Proof: See [Hu].

Extension of a lift from subcomplex. Let $p : X \rightarrow Y$ be a fibration with fiber F with a Postnikov tower $\{X_n, q_n\}$ and let (W, A) be a CW-complex and let $i : A \rightarrow W$ be the canonical inclusion. Suppose that we have a map $f : W \rightarrow Y$ and a lift $F' : A \rightarrow X$ such that $p \circ F' = f \circ i$. The picture is given in the following diagram

$$\begin{array}{ccccc} & & \cdots & & \\ & & \downarrow q_3 & & \\ & & X_2 & \longrightarrow & K(\pi_2(F), 3) \\ & \nearrow p_3 & \downarrow q_2 & & \\ A & \xrightarrow{F'} X & \xrightarrow{p_1} X_1 & \longrightarrow & K(\pi_1(F), 2) \\ & \searrow p & \downarrow q_1 & & \\ W & \xrightarrow{f} Y & \longrightarrow & K(\pi_0(F), 1). \end{array} \quad (50)$$

Lemma 11. *Let us keep the notation as above. Suppose that $H^{i+1}(W, A, \pi_i(F)) = 0$ for all $i \geq 0$. Then there exist a lift $F : W \rightarrow X$ such that $F \circ i = F'$ and $p \circ F = f$.*

Proof: Suppose that we have a lift $F_i : W \rightarrow X_i$ such that $q_1 \circ \dots \circ q_i \circ F_i = f$ and $p_i \circ F' = F_i \circ i$. Then the obstruction of lifting F_i to $F_{i+1} : W \rightarrow X_{i+1}$ such that $q_{i+1} \circ F_{i+1} = F_i$ and $p_{i+1} \circ F' = F_{i+1} \circ i$ is a class $\omega_i \in H^{i+1}(W, A, \pi_i(F))$. Thus if $\omega_i = 0$ for all i , we can find an extension $W \rightarrow \lim_{\leftarrow} X_i$ and further to $W \rightarrow X$. Full discussion is in [H]. \square

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